References

[1]Widmayer, E., Clevenson, S. A., and Leadbetter, S. A., "Some Measurements of Aerodynamic Forces and Moments at Subsonic Speeds on a Rectangular Wing of Aspect Ratio 2 Oscillating About the Midchord," NASA TN 4240, 1958.

[2]Davis, D. D. and Moore, D., "Analytical Study of Blockage and Lift-Interference Corrections for Slotted Tunnels Obtained by the Substitution of an Equivalent Homogeneous Boundary for the Discrete Slots," NASA Research Memo L53E07b, June 1953.

[3]Acum, W. E. A., "A Simplified Approach to the Phenomenon of Wind Tunnel Resonance," Aeronautical Research Council Reports and Memoranda No. 3371, 1962.

[4]Lee, I., "Resonance Prediction for Slotted Wind Tunnel by the Finite Element Method," AIAA Paper 86-0898, May 1986.

[5]Tijdeman, H., "On the Propagation of Sound Waves in Cylindrical Tubes," *Journal of Sound and Vibration*, Vol. 39, March 1975, pp. 1-33.

[6]Craggs, A., "A Finite Element Method for Damped Acoustic Systems: An Application to Evaluate the Performance of Reactive Mufflers," *Journal of Sound and Vibration*, Vol. 48, Oct. 1976, pp. 377-392.

[7]Kagawa, Y., Yamabuchi, T., and Mori, A., "Finite Element Simulation of an Axisymmetric Acoustic Transmission System with a Sound Absorbing Wall," *Journal of Sound and Vibration*, Vol. 53, Aug. 1977, pp. 357-374.

[8]Joppa, P. D. and Fyfe, I. M., "A Finite Element Analysis of the Impedance Properties of Irregular Shaped Cavities with Absorptive Boundaries," *Journal of Sound and Vibration*, Vol. 56, Jan. 1978, pp. 61-69.

[9]Gladwell, G. M. L., "A Variational Formulation of Damped Acousto-Structural Vibration Problems," *Journal of Sound and Vibration*, Vol. 4, Sept. 1966, pp. 172-186.

Iterative Methods for Design Sensitivity Analysis

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Introduction

ITERATIVE methods are presented for obtaining design sensitivity coefficients (or derivatives) of implicit functions. Design derivatives are important not only in gradient-based optimization codes, but also for examining tradeoffs, system identification, and probabilistic design, to name a few. Iterative methods are presented for both the algebraic and eigenvalue problems; stress, eigenvalue, and eigenvector derivatives are considered. The iterative approaches provide approximate derivatives. They are very simple to implement in a program, even for complex structural response. Iterative methods for sensitivity, for a class of structural problems, were suggested in Ref. 1. General papers on reanalysis schemes may be found in Refs. 2 and 3.

In the next section, iterative methods for sensitivity of displacement and stress are developed. Following this, eigenvalue and eigenvector sensitivity is considered.

Displacement and Stress Sensitivity

The problem of obtaining design derivatives of displacements and stresses, for a finite-element model of the structure, is considered. Consider a function $g = g(b, z)$. This represents a stress constraint, with $b = (k \times 1)$ design vector and $z = (n \times 1)$ displacement vector, which is obtained from the finite-element equations $K(b)\ z = F(b)$, where K is an $(n \times n)$ structural stiffness matrix, and F is an $(n \times 1)$ nodal load vector. Let b^0 be the current design. At this stage, the analysis has been completed. Thus, the decomposed $K(b^0)$ and z^0 are known. The derivative of the function g with respect to the ith design variable is given by

$$\frac{\mathrm{d}g}{\mathrm{d}b_i} = \frac{\partial g}{\partial b_i} + \frac{\partial g}{\partial z} \cdot \frac{\mathrm{d}z}{\mathrm{d}b_i} \tag{1}$$

The partial derivatives $\partial g/\partial b$ and $\partial g/\partial z$ are readily available using the finite-element relations. The problem, therefore, is to compute the displacement sensitivity $\mathrm{d}z/\mathrm{d}b$. An iterative approach for computing this is now given.

Corresponding to the ith design variable, let the perturbed design vector b^ϵ be defined as

$$b^\epsilon = (b_1^0, b_2^0, \ldots, b_i^0 + \epsilon, \ldots, b_k^0)^T \tag{2}$$

The perturbation ϵ is relatively small, and a value of 1% of b_i is suggested. The problem is to find z^ϵ, the solution of

$$K(b^\epsilon)\ z^\epsilon = F(b^\epsilon) \tag{3}$$

using the decomposed $K(b^0)$ and z^0. A modified version of the residual-correction scheme given in Ref. 3 is given below.

Algorithm 1 (Displacement and Stress Sensitivity)

Step 0: Set $j = 0$. Choose the perturbation ϵ and the error tolerance Δ. Define b^ϵ as in Eq. (2).

Step 1: Calculate the residual r^j from

$$r^j = K(b^\epsilon)\ z^j - F(b^\epsilon) \tag{4}$$

Step 2: Solve for the correction e^j from

$$K(b^0)\ e^j = -r^j \tag{5}$$

Step 3: Update $z^{j+1} = z^j + e^j$

Step 4: Check the convergence criterion

$$||z^{j+1} - z^j|| \le \Delta \tag{6}$$

If satisfied, then set $z^\epsilon = z^{j+1}$, and compute the displacement sensitivity as

$$\frac{\mathrm{d}z}{\mathrm{d}b_i} \approx \frac{z^\epsilon - z^0}{\epsilon} \tag{7}$$

The stress sensitivity can be recovered from Eq. (1). Otherwise, set $j = j + 1$ and re-execute steps 1–4.

Numerical results and comparison with the exact and semianalytical methods discussed in the literature are presented subsequently. Theoretically, it can be shown that the above scheme will converge provided[3]

$$r_\sigma[I - K^{-1}(b^0)\ K(b^\epsilon)] < 1 \tag{8}$$

where $r_\sigma(A)$ = spectral radius of the matrix A, which is the maximum size of the eigenvalues of A. In the problem considered here, $K(b^0)$ and $K(b^\epsilon)$ are roughly equal because ϵ is small, and Eq. (8) can generally be expected to hold.

Eigenvalue and Eigenvector Sensitivity

Eigenvalue sensitivity is useful when resonant frequencies or critical buckling loads need to be restricted. Exact analytical expressions for eigenvalue sensitivity can be readily derived for the case of nonrepeated roots.[4] The problem of obtaining eigenvector sensitivity, on the other hand, is more complicated

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and is an area of current interest (e.g., Ref. 5). Eigenvector sensitivity is useful in obtaining the design derivatives of forced dynamic response. Here, an iterative approach is presented for approximate derivatives of eigenvalues and eigenvectors. The approach is particularly easy to implement in a program and provides both eigenvalue and eigenvector derivatives simultaneously. Furthermore, the derivative of a particular eigenvector does not require knowledge of all eigenvectors of the problem, as with certain analytical methods.

Consider the generalized eigenvalue problem

$$K(\boldsymbol{b})\boldsymbol{y} = \lambda \boldsymbol{M}(\boldsymbol{b})\boldsymbol{y} \tag{9}$$

where λ is a particular nonrepeated eigenvalue, and $\boldsymbol{y}$ is the associated eigenvector. For the frequency problem, $\boldsymbol{K}$ and $\boldsymbol{M}$ represent the structural stiffness and mass matrices, respectively. It is desired to find the sensitivities $\mathrm{d}\lambda/\mathrm{d}\boldsymbol{b}$ and $\mathrm{d}\boldsymbol{y}/\mathrm{d}\boldsymbol{b}$. Let $\boldsymbol{b}^0$ be the current design vector and $(\lambda_0, \boldsymbol{y}^0)$ be a given eigenvalue-eigenvector pair at the current design. Let $\boldsymbol{b}^\epsilon$ be a perturbed design vector. The residual is given by

$$\boldsymbol{R} = \boldsymbol{K}(\boldsymbol{b}^\epsilon)\boldsymbol{y}^\epsilon - \lambda_\epsilon \boldsymbol{M}(\boldsymbol{b}^\epsilon)\boldsymbol{y}^\epsilon \tag{10}$$

The object is to solve the nonlinear equations $\boldsymbol{R} = \boldsymbol{0}$ for the unknowns λ_ϵ and $\boldsymbol{y}^\epsilon$. The Newton-Raphson technique is used for this purpose. The Jacobian J of the system in Eq. (10) is $[\partial \boldsymbol{R}/\partial \boldsymbol{y}^\epsilon, \partial \boldsymbol{R}/\partial \lambda_\epsilon]$. This, together with the normalization condition $-\boldsymbol{y}^T \boldsymbol{M}\, \delta \boldsymbol{y} = 0$, and preserving constant coefficient matrices, leads to solving the system

$$[C]\begin{pmatrix} \delta y \\ \delta\lambda \end{pmatrix} = \begin{pmatrix} -R \\ 0 \end{pmatrix} \tag{11}$$

where

$$C = \begin{pmatrix} K(b^0) - \lambda_0 M(b^0) & -M(b^0)y^0 \\ -y^{0T} M(b^0) & 0 \end{pmatrix} \tag{12}$$

The coefficient matrix is symmetric and nonsingular for the case of nonrepeated roots.[6] The algorithm for eigenvalue-eigenvector sensitivity is now given.

Algorithm 2 (Eigenvalue-Eigenvector Sensitivity)

Step 0: Set $j = 0$. Choose the perturbation ϵ and the error tolerances Δ_1 and Δ_2. Define $\boldsymbol{b}^\epsilon$. Decompose the matrix $\boldsymbol{C}$ given in Eq. (12).

Step 1: Define the residual $\boldsymbol{R}^j = \boldsymbol{K}(\boldsymbol{b}^\epsilon)\boldsymbol{y}^j - \lambda_j \boldsymbol{M}(\boldsymbol{b}^\epsilon)\boldsymbol{y}$.

Step 2: Solve Eq. (11).

Step 3: Update $y^{j+1} = y^j + \delta y$ and $\lambda_{j+1} = \lambda_j + \delta\lambda$.

Step 4: Check the convergence criterion

$$\|\delta y\| \le \Delta_1, \qquad |\delta\lambda| \le \Delta_2 \tag{13}$$

If satisfied, then set $y^\epsilon = y^{j+1}$, $\lambda_\epsilon = \lambda_{j+1}$, and compute the sensitivity as

$$\frac{\mathrm{d}y}{\mathrm{d}b_i} \approx \frac{y^\epsilon - y^0}{\epsilon}, \qquad \frac{d\lambda}{\mathrm{d}b_i} \approx \frac{\lambda_\epsilon - \lambda_0}{\epsilon} \tag{14}$$

Otherwise, set $j = j + 1$ and re-execute steps 1–4 above.

Numerical Results

Two examples are presented below. The first deals with stress sensitivity in a thin plate with inverse thicknesses as design variables. The relative merits of the iterative approach with the semianalytical method are discussed. The second example deals with the eigenvalue-eigenvector sensitivity of a planar frame.

Thin-Plate Program

Consider the plane stress problem in Fig. 1, where inverse thicknesses are the design variables. That is, the reciprocal of the plate thickness is chosen as a design variable. Inverse design variables are used in optimal design because they tend to linearize the stress function and lead to improved convergence. The stress constraint function is the von Mises failure criterion in the element. Constant strain triangular elements are used. For brevity, only the design sensitivity coefficients, $\mathrm{d}g_{19}/\mathrm{d}b_{19}$ and $\mathrm{d}g_{24}/\mathrm{d}b_{24}$, are presented in Table 1. The sensitivity vectors have been obtained using Algorithm 1 discussed earlier. In Table 1, the results obtained by the iterative method are compared with the semianalytical method used widely in the literature, based on the formula

$$K\frac{\mathrm{d}z}{\mathrm{d}b_i} = -\left(\frac{K(b^\epsilon) - K(b^0)}{\epsilon}\right)z^0 + \frac{[F(b^\epsilon) - F(b^0)]}{\epsilon} \tag{15}$$

It is interesting to note that the semianalytical method yields the same result as the first iteration of the iterative method. However, the iterative method further improves upon this and approaches the exact sensitivity (Table 1). Since all methods yield values of acceptable accuracy, use of the iterative method for this type of problem is not any more advantageous. The comparison, however, serves to illustrate the nature of the iterative process. It is noted that, when using direct variables (as opposed to reciprocal variables), the semianalytical method yields essentially exact sensitivity because stiffness is a linear function of design variables. In this case, only one iteration of the iterative method needs to be performed.

Plane-Frame Problem

Consider the frame structure in Fig. 2. The design variables associated with the I-section are $\boldsymbol{b} = (h, w, t_w, t_f)$, as shown. The problem has a total of 24 design variables and 12 degrees of freedom. The sensitivity of the lowest eigenvalue and corresponding eigenvector obtained using Algorithm 2 is evaluated. The iterative method takes 0.99 s of CPU time, compared to 1.72 s taken by Newton's forward-difference formula. Thus, for larger problems, the iterative method will provide a more efficient alternative to analytical sensitivities

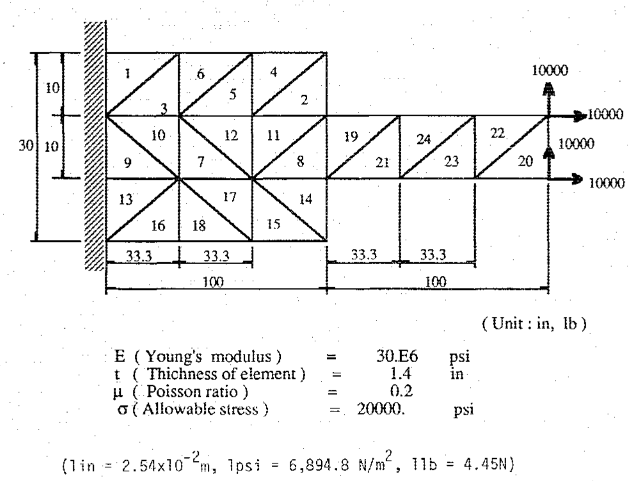

Fig. 1 Thin-plate problem.

Table 1 Stress sensitivity coefficients for thin-plate problem

Method		$\frac{\mathrm{d}g_{19}}{\mathrm{d}b_{19}}$	$\frac{\mathrm{d}g_{24}}{\mathrm{d}b_{24}}$
Iterative	1	8.7098	5.6437
	2	8.7949	5.6980
	3	8.7957	5.6986
Semianalytical		8.7098	5.6437
Exact		8.7969	5.7002

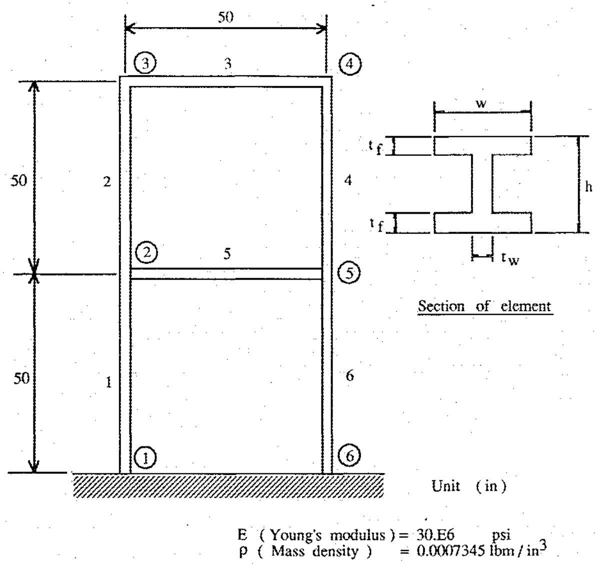

Fig. 2 Plane-frame problem.

as compared with pure divided-difference schemes. The maximum number of iterations required for an error tolerance of 10^{-7} is five, which indicates rapid convergence. Also, the algorithm does not require computation of all eigenvectors to find the sensitivity of a few specific eigenvectors.

Summary and Conclusions

A numerical method has been presented for design sensitivity analysis. The idea is based on using iterative methods for reanalysis of the structure due to a small perturbation in the design variable. A forward-difference scheme then yields the approximate sensitivity. Algorithms for displacement and stress sensitivity as well as for eigenvalues and eigenvector sensitivity are developed. The iterative schemes have been modified so that the coefficient matrices are constant and hence decomposed only once. The convergence is found to be very rapid. Implementation of the algorithms is found to be simple. The method can extend to eigenvalue sensitivity of problems with repeated roots. This extension is also important to avoid ill conditioning of the coefficient matrix in the vicinity of bifurcation points, which occurs when nonlinear structural response is considered.

References

[1]Adelman, H. M., Haftka, R. T., Camarda, C. J., and Walsh, J. L., "Structural Sensitivity Analysis: Methods, Applications and Needs," *Proceedings of the Symposium on Recent Experiences in Multidisciplinary Analysis and Optimization*, NASA Langley Research Center, Hampton, VA, April 1984, pp. 367–384.

[2]Arora, J. S., "Survey of Structural Reanalysis Techniques, "*Journal of the Structural Division, ASCE*, Vol. 102, No. ST4, Proceedings Paper 12056, April 1976, pp. 783–802.

[3]Atkinson, K. E., *An Introduction to Numerical Analysis*, Wiley, New York, 1978.

[4]Haug, E. J. and Arora, J. S., *Applied Optimal Design*, Wiley, New York, 1979.

[5]Ojalvo, I. U., "Efficient Computation of Mode-Shape Derivatives for Large Dynamic Systems," *AIAA Journal*, Vol. 25, Oct. 1987, pp. 1386–1390.

[6]Robinson, A. R. and Harris, J. F., "Improving Approximate Eigenvalues and Eigenvectors," *Journal of the Engineering Mechanics Division*, Vol. 97, No. EM2, April 1971, pp. 457–475.

Higher-Order Finite Element for Short Beams

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Introduction

PROBLEMS involving relatively short beams are common in some aerospace structures. In such beams, the transverse shear has an important effect on the deformations and stresses. The elementary Bernoulli-Euler beam theory, which does not include shear deformation, is inadequate. An early theory that includes transverse shear was presented by Timoshenko[1] in 1921. This theory assumed that plane sections remain plane during the deformation but not necessarily normal to the middle surface. Since then investigators have attempted improvement by modifying the shear constant in Timoshenko's theory.[2] More recently, a number of papers have appeared which offer alternate approaches. In 1981, Levinson[3] introduced a new beam theory that included warping of the cross section and imposed a shear-free condition on the top and bottom surfaces. Rehfield and Murthy,[4] in 1982, solved a simply supported beam problem using a modified plane stress elasticity solution to include transverse shear and normal strains. Their approach differs from Timoshenko's because of assumptions regarding the stresses rather than the strains. In 1984, Murty[5] presented another higher-order beam theory. In 1986, Suzuki[6] presented a theory for short beams also, using assumptions about the stress distribution. Some articles have appeared which utilize finite elements with the shear effect included.[7-12] Most of these removed the restriction that cross sections normal to the undeformed middle surface of the beam remain normal during the deformation (similar to the Timoshenko theory). These normal cross sections, however, do remain plane. The finite element presented here removes this restriction by allowing the cross sections to deform into a shape that can be described by a function that includes quadratic and cubic terms as well as the linear one. Although these additional terms create a more complex element (six more degrees of freedom), the resulting element is shown to provide good results for both stresses and displacements when compared with other theories for short beams. Finite elements are, of course, easily extrapolated to practical problems with more complex loading and boundary conditions.

Formulation

The proposed beam finite element is shown in Fig. 1a. It has length ℓ and a cross section symmetric about the y axis. The x and y displacements of a typical point in the beam are u and v, respectively. The x and y displacements of the corresponding point on the middle surface of the beam are U and V. The rotation of the cross section is characterized by the three terms, Φ_1, Φ_2, and Φ_3. The expressions for u and v in terms of the five functions U, V, Φ_1, Φ_2, and Φ_3 are

$$u = U - \Phi_1 y - \Phi_2 y^2 - \Phi_3 y^3, \quad v = V \tag{1}$$

where U, V, Φ_1, Φ_2, and Φ_3 are functions of x (the shape functions) to be described below in Eqs. (4). The axial strain ϵ_x and the shear strain γ_{xy} are then

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